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| RESEARCH ARTICLE

NEW MULTIPLE SOLUTIONS FOR SOME PERIODIC BOUNDARY VALUE PROBLEMS WITH ψ-LAPLACIAN

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| ABSTRACT
We study the existence of multiple solutions of the quasilinear equation
$$(\psi(u'(t)))' = f(t,u(t),u'(t)), \ \ t \in [0,T]$$
submitted to periodic boundary conditions, where $\psi:]-a,a[\to \mathbb{R}$, with $0 < a < +\infty$, is an increasing homeomorphism such that $\psi(0) = 0$. Combining some sign conditions and lower and upper solutions method, we obtain existence of two or Three solutions.

| KEYWORDS

$\psi-$ Laplacian; L^1 $-$Carathéodory function; nonlinear Neumann-Steklov problem; Periodic problem; Lower and upper-solutions; sign conditions.

| ARTICLE INFORMATION

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1. Introduction

This work is devoted to the study of the existence of solutions of the the quasilinear equation

$$\left(\psi(u'(t))\right)' = f(t,u(t),u'(t)), \qquad \forall \ \ t \in [0,T] \tag{1}$$

submitted to periodic boundary conditions

$$u'(0) = u'(T), \qquad\qquad u(0) = u(T). \tag{2}$$

Where $\psi:]-a,a[\to \mathbb{R}$ with $0 < a < +\infty$, is an increasing homeomorphism such that $\psi(0) = 0$.

In section 2, we give some preliminaries results.

In section 3, combining some sign conditions and existence only one strict lower solution and one strict upper solution of problem (1)-(2), we prove existence of at least two or three solutions of problem (1)-(2). We show in this section that the existence of at least two or three solutions for certain forced second order quasilinear equations submitted to periodic boundary conditions is guaranteed by the presence of one strict lower solution and one strict upper solution.

2. Methodology

Generally, in the lower and upper solutions method, to show existence of at least one solution of a problem, we need existence of at least one lower solution and at least one upper solution. In the case of the sign conditions method, we usually need two sign conditions to show existence of at least one solution of a problem.

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In (Goli & Adjé, 2016), the authors proved existence of solutions of (1)-(2), when there exists only one sign condition and only one lower solution or only one upper solution.

We use the results proven by Goli and Adjé to show:

- Existence of at least two solutions of (1)-(2), when we have only one sign condition, one strict lower solution and one strict upper solution.
- Existence of at least three solutions of (1)-(2), when we have two sign conditions, one strict lower solution and one strict upper solution.
- For some periodic problems, the existence of two real numbers a and b such that $a > b$ and $f(t, a, 0) < 0$ and $f(t, b, 0) > 0$, $\forall t \in [0, T]$, allows us to affirm the existence of 2 or 3 solutions.

3. Preliminary

Definition 3.1. A solution of problem (1)-(2) is a function $u \in C^1([0,T])$ such that $\psi(u') \in C^1([0,T])$, $||u'||_\infty < a$ and satisfies (1)-(2).

Definition 3.2. A function $\delta \in C^1([0,T])$ is a lower-solution of the problem (1)-(2) if $||\delta\,'||_\infty < a$, $\psi(\delta\,') \in C^1([0,T])$,

$$\left(\psi\left(\delta'(t)\right)\right)' \geq f\left(t, \delta(t), \delta\,'(t)\right), \quad t \in [0,T], \tag{3}$$

$$\delta'(0) \geq \delta'(T) \;\; and \;\; \delta(0) = \delta(T). \tag{4}$$

Definition 3.3. A function $\gamma \in C^1([0,T])$ is an upper-solution of the problem (1)-(2) if $||\gamma\,'||_\infty < a$, $\psi(\gamma\,') \in C^1([0,T])$,

$$\left(\psi\left(\gamma'(t)\right)\right)' \leq f\left(t, \gamma(t), \gamma\,'(t)\right), \quad t \in [0,T], \tag{5}$$

$$\gamma\,'(0) \leq \gamma\,'(T) \;\; and \;\; \gamma\,(0) = \gamma\,(T). \tag{6}$$

Definition 3.4. A lower-solution δ of (1)-(2) is said to be strict if every solution u of (1)-(2) with $u(t) \geq \delta(t)$ on $[0,T]$ is such that $u(t) > \delta(t)$ on $[0,T]$.

Definition 3.5. A upper-solution γ of (1)-(2) is said to be strict if every solution u of (1)-(2) with $u(t) \leq \gamma(t)$ on $[0,T]$ is such that $u(t) < \gamma(t)$ on $[0,T]$.

Remark 3.1.

- A lower solution of (1)-(2) is strict if the inequality (4) is strict for all $t \in [0,T]$;
- An upper solution of (1)-(2) is strict if the inequality (6) is strict for all $t \in [0,T]$.

Theorem 3.1. Assume that:

1. there exists a lower-solution δ of the problem (1)-(2);
2. $\exists\, R > 0$ such that

$$u_L \geq R \;and\; \left||u'|\right|_\infty < a \;\Rightarrow \int_0^T f\left(t, u(t), u'(t)\right)dt > 0. \tag{7}$$

Then the problem (1)-(2) admits at least one solution u such that $\delta(t) \leq u(t)$ for all $t \in [0,T]$.

Proof. See Theorem 3.2 and its proof in (Goli & Adjé, 2016).

Theorem 3.2. Assume that:

1. there exists an upper-solution γ of the problem (1)-(2);
2. $\exists\, R > 0$ such that

$$u_M \leq -R \;and\; \left||u'|\right|_\infty < a \;\Rightarrow \int_0^T f\left(t, u(t), u'(t)\right)dt < 0. \tag{8}$$

Then the problem (1)-(2) admits at least one solution u such that $u(t) \leq \gamma(t)$ for all $t \in [0,T]$.

Proof. See (Goli & Adjé, 2016).

Theorem 3.3. Assume that there exist a lower-solution δ and an upper-solution γ of (1)-(2) such that

$$\exists \tilde{t} \in [0,T] \text{ such that } \delta(\tilde{t}) > \gamma(\tilde{t});$$

Then the problem (1)-(2) admits at least one solution u, such that

$$\min\{\delta(t_u), \gamma(t_u)\} \leq u(t_u) \leq \max\{\delta(t_u), \gamma(t_u)\}$$

for some $t_u \in [0,T]$ and

$$||u||_\infty \leq \max\left\{||\delta||_\infty, ||\gamma||_\infty\right\} + aT.$$

Proof. See Theorem 1 and its proof in (Bereanu and Al, 2013).

4. Existence of multiple solutions

a. Existence of at least two solutions

Theorem 4.1. Assume that:

1. there exist a strict lower-solution δ and a strict upper-solution γ of (1)-(2) such that
$$\exists\, t \in [0,T], \quad \delta(t) > \gamma(t);$$
2. $\exists\, R > 0$ such that

$$u_L \geq R \text{ and } ||u'||_\infty < a \Rightarrow \int_0^T f(t, u(t), u'(t))dt > 0. \tag{9}$$

Then the problem (1)-(2) admits at least two solutions u and w such that:

- $\delta(t) < u(t) \text{ for all } t \in [0,T]$.
- $\gamma(t_w) \leq u(t_w) \leq \delta(t_w)$ for some $t_w \in [0,T]$.

Proof. By Theorem 3.1. and the fact that δ is strict, the problem (1)-(2) admits at least one solution u such that

$$\delta(t) < u(t) \text{ for all } t \in [0,T]. \tag{10}$$

Using the Theorem 3.3, the problem (1)-(2) admits at least one solution w such that

$$\gamma(t_w) = \min\{\delta(t_w), \gamma(t_w)\} \leq w(t_w) \leq \max\{\delta(t_w), \gamma(t_w)\} = \delta(t_w). \tag{11}$$

Using (10) and (11), we have $u \neq w$.

Theorem 4.2. Assume that:

1. there exist a strict lower-solution δ and a strict upper-solution γ of (1)-(2) such that
$$\exists\, t \in [0,T], \quad \delta(t) > \gamma(t);$$
2. $\exists\, R > 0$ such that

$$u_M \leq -R \text{ and } ||u'||_\infty < a \Rightarrow \int_0^T f(t, u(t), u'(t))dt < 0. \tag{12}$$

Then the problem (1)-(2) admits at least two solutions v and w such that:

- $v(t) < \gamma(t) \text{ for all } t \in [0,T]$;
- $\gamma(t_w) \leq u(t_w) \leq \delta(t_w)$ for some $t_w \in [0,T]$.

Proof. By Theorem 3.2. and the fact that γ is strict, the problem (1)-(2) admits at least one solution v such that

$$v(t) < \gamma(t) \text{ for all } t \in [0,T]. \tag{13}$$

Using the Theorem 3.3., the problem (1)-(2) admits at least one solution w such that

$$\gamma(t_w) = \min\{\delta(t_w), \gamma(t_w)\} \leq w(t_w) \leq \max\{\delta(t_w), \gamma(t_w)\} = \delta(t_w). \tag{14}$$

Using (13) and (14), we have $v \neq w$.

Corollary 4.1. Assume that:

1.
$$\lim_{u\to-\infty} f(t,u,v) = -\infty \;\; or \;\; \lim_{u\to+\infty} f(t,u,v) = +\infty$$
uniformly in $\{(t,v);\ (t,v) \in [0,T]\times[-a,a]\}$;
2. there exist a strict lower-solution δ and a strict upper-solution γ of (1)-(2).

Then the problem (1)-(2) has at least two solutions.

Corollary 4.2. Assume that:

1.
$$\lim_{u\to-\infty} f(t,u,v) = -\infty \;\; or \;\; \lim_{u\to+\infty} f(t,u,v) = +\infty$$
uniformly in $\{(t,v);\ (t,v) \in [0,T]\times[-a,a]\}$;
2. there exist $\delta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that $\delta > \gamma$ and $f(t,\delta,0) < 0$ and $f(t,\gamma,0) > 0$, $\forall\, t \in [0,T]$.

Then the problem (1)-(2) has at least two solutions.

Example 4.1. Consider the problem

$$\left(\frac{u'(t)}{\sqrt{1-\left(u'(t)\right)^2}}\right)' = t^2 + (\mathrm{u(t)})^2 + -9 + t^4(\mathrm{u'(t)} + \cos(\mathrm{u(t)}))\,,\;\; t \in [0,1],$$
$$u'(0) = u'(1), \qquad u(0) = u(1),$$

We can take $\delta = 1$ and $\gamma = -10$.
We have:
$$\lim_{u\to+\infty} f(t,u,v) = +\infty$$
uniformly in $\{(t,v);\ (t,v) \in [0,T]\times[-a,a]\}$,
$$f(t,\delta,0) \;=\; f(t,1,0) = t^2 + (1)^2 - 9 + t^4(0+\cos(1))$$
$$= t^2 - 8 + t^4(0+\cos(1)) \;< 0,$$
$$f(t,\gamma,0) \;=\; f(t,1,0) = t^2 + (-10)^2 - 9 + t^4(0+\cos(1))$$
$$= t^2 + 91 + t^4(0+\cos(-10)) \;< 0.$$
Using Corollary 3.2, we deduce the existence of at least two solutions.

b. Existence of at least three solutions

Theorem 4.3. Assume that:

1. there exist a strict lower-solution δ and a strict upper-solution γ of (1)-(2) such that
$$\exists\, t \in [0,T], \qquad \delta(t) > \gamma(t);$$
2. $\exists\, R > 0$ such that
$$u_L \geq R \; and \; \left|\left|u'\right|\right|_\infty < a \;\Rightarrow \int_0^T f\big(t,u(t),u'(t)\big)dt > 0$$
and

$\exists\, R_1 > 0$ such that
$$u_M \leq -R_1 \; and \; \left|\left|u'\right|\right|_\infty < a \;\Rightarrow \int_0^T f\big(t,u(t),u'(t)\big)dt < 0.$$

Then the problem (1)-(2) admits at least three solutions u, v and w such that:

- $\delta(t) < u(t)\;\; for\ all\ t \in [0,T]$;
- $v(t) < \gamma(t)\;\; for\ all\ t \in [0,T]$;
- $\gamma(t_w) \leq u(t_w) \leq \delta(t_w)$ for some $t_w \in [0,T]$.

Proof. By Theorem 3.1. and the fact that δ is strict, the problem (1)-(2) admits at least one solution u such that

$$\delta(t) < u(t) \ \ for\ all\ \ t \in [0,T]. \tag{15}$$

By Theorem 3.2. and the fact that γ is strict, the problem (1)-(2) admits at least one solution v such that

$$v(t) < \gamma(t) \ \ for\ all\ \ t \in [0,T]. \tag{16}$$

Using the Theorem 3.3., the problem (1)-(2) admits at least one solution w such that

$$\gamma(t_w) = \min\{\delta(t_w), \gamma(t_w)\} \leq w(t_w) \leq \max\{\delta(t_w), \gamma(t_w)\} = \delta(t_w). \tag{17}$$

Using (15), (16) and (17), we have $u \neq v$, $u \neq w$ and $v \neq w$.

Corollary 4.3. Assume that:

1.
$$\lim_{u\to-\infty} f(t,u,v) = -\infty \ \ and \ \lim_{u\to+\infty} f(t,u,v) = +\infty$$
uniformly in $\{(t,v);\ (t,v) \in [0,T] \times [-a,a]\}$;
2. there exist a strict lower-solution δ and a strict upper-solution γ of (1)-(2).

Then the problem (1)-(2) has at least three solutions.

Corollary 4.4. Assume that:

1.
$$\lim_{u\to-\infty} f(t,u,v) = -\infty \ \ and \ \lim_{u\to+\infty} f(t,u,v) = +\infty$$
uniformly in $\{(t,v);\ (t,v) \in [0,T] \times [-a,a]\}$;
2. there exist $\delta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that $\delta > \gamma$ and $f(t,\delta,0) < 0$ and $f(t,\gamma,0) > 0$, $\forall\, t \in [0,T]$.

Then the problem (1)-(2) has at least three solutions.

Example 4.2.

$$\left(\frac{u'(t)}{\sqrt{1-\left(u'(t)\right)^2}}\right)' = \frac{t}{3} + (\mathrm{u(t)})^3 - 12u(t) - 1 + \sin t^4\,(\mathrm{u'(t)} + \arctan(\mathrm{u(t)}))\,, \ \ t \in \ [0,1],$$

$$u'(0) = u'(1), \qquad u(0) = u(1),$$

We can take $\delta = 2$ and $\gamma = -2$.

We have:

$$\lim_{u\to-\infty} f(t,u,v) = -\infty \ \ and \ \lim_{u\to+\infty} f(t,u,v) = +\infty$$

uniformly in $\{(t,v);\ (t,v) \in [0,T] \times [-a,a]\}$,

$f(t,\delta,0) \ = \ f(t,2{,}0) = \frac{t}{3} - 17 + \sin t^4\,(0 + \arctan(2)) \ < 0,$

$f(t,\gamma,0) = \ f(t,-2{,}0) = \frac{t}{3} + 15 + \sin t^4\,(0 + \arctan(-2)) > 0.$

Using Corollary 3.4, we deduce the existence of at least three solutions.

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