Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

A Study on the Relationships Between the Norms of Complex Polynomials of Degrees 2, 3, and 4 and Their Derivatives Belonging to a_2

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| ABSTRACT
In this paper, relations between the norms of complex polynomials of degrees 2, 3, and 4 and their derivatives are studied. Using bound-preserving convolution operators and interpolation formulas, we derive inequalities governing these polynomial norms. Clement Frappier previously found a relation for $n \geq 6$, but for $n = 2, 3, 4$, a unique relation does not exist. We establish new bounds for these cases by employing determinant analysis and principal minor calculations. The theoretical framework is constructed using Hermitian matrices, semi-bilinear functions, and norm-preserving operators, leading to a refined approach for identifying the smallest positive roots of characteristic equations. The results provide a deeper understanding of polynomial inequalities and contribute to the broader study of functional analysis and complex function theory.

| KEYWORDS

Positive Definiteness, Hadamard Product, Hermitian Matrix, Semi-Bilinear, Holomorphic Function.

| ARTICLE INFORMATION

ACCEPTED: 18 June 2025 **PUBLISHED:** 24 Jul 2025 **DOI:** 10.32996/jmss.2025.6.3.4

1. Introduction

In this paper relationships between the norms of the polynomials of degree 2, 3, 4 and its derivatives which belong to a_2 are studied. The proof methods are "bound preserving convolution operators in the unit disk and interpolation formulas". By the help of theorem.1 and definition ofB_n^0, principle"$Q(z) \in B_n^0$ then $\|Q * p\| \leq \|p\|$" and $Q \in B_n^0 \Leftrightarrow Q * \in B_n^0$, find a related polynomial for every inequality $Q \in B_n^0$ such that, the conditions of the theorem satisfy. For this purpose, we need to calculate and show that the related coefficients determinant of $Q(z)$, in which area this determinant and its principal minors are positive. For the destining of this polynomial $Q \in B_n^0$ the theorem 1 and theorem 7 are used.

In this paper we will use these abbreviations:
$\boldsymbol{A}$ Be set of all holomorphic functions in$|z| < 1$,
$\boldsymbol{A_0}$ All function $f \in A$ with $f(0) = 1$,
R Set of all functions with $Re(f(z)) > \frac{1}{2}$.
$\overline{co}$ Convex domain.

2. Literature Review

Recent advancements in the study of polynomial norms and their derivatives have yielded significant insights, particularly concerning integral-norm estimates and the behavior of polynomials with restricted zeros.
In 2022, Mir and Bidkham established integral-norm estimates for lacunary-type polynomials in the complex plane. Their research extended classical Bernstein-type inequalities by relating the $L^{\gamma} - norm$ of the polar derivative to that of the polynomial itself. This

work generalized existing estimates and provided a deeper understanding of the relationship between polynomials and their polar derivatives. (Mir & Bidkham, 2022)
Furthermore, Bhat, Wani, and Rather (2024) explored upper bound estimates for the modulus of polynomial derivatives within the unit disk, considering the positioning of zeros and extremal coefficients. They extended their findings to the polar derivative of polynomials, sharpening and generalizing Erdös–Lax type inequalities. Their results offer enhanced estimates that account for zero restrictions, contributing to the refinement of polynomial inequality theories. (Bhat, Wani, & Rather, 2024)
Additionally, Ahmadi, de Klerk, and Hall (2018) investigated polynomial norms, focusing on norms that are the dth root of a degree-d homogeneous polynomial. They provided necessary and sufficient conditions for such norms and demonstrated that any norm can be approximated arbitrarily well by a polynomial norm. Their study also addressed the computational complexity of testing whether a form gives a polynomial norm, highlighting the NP-hardness of this problem for degree-4 forms. (Ahmadi, Klerk, & Hall, 2018)
These contemporary studies collectively enhance the theoretical framework surrounding polynomial norms and their derivatives, offering refined inequalities and computational insights that are pertinent to ongoing research in complex analysis and polynomial theory.

3. Mathematical Foundations and Polynomial Norm Analysis

This section presents key definitions, theorems, and mathematical concepts essential for analyzing polynomial norms and their derivatives. Fundamental properties of Hermitian matrices, semi-bilinear functions, and bound-preserving convolution operators are introduced to establish a foundation for the study. The paper employs determinant-based verification techniques and principal minor calculations to derive optimal polynomial norm inequalities. By leveraging these mathematical tools, we systematically explore the relationships between polynomial norms and their derivatives for degrees 2, 3, and 4. Additionally, we provide an analytical approach to determining the smallest positive roots that define the best possible bounds for these norms.

Definition 1. Let $f(z) = \sum_{i=1}^{n} a_k z^k$ and $g(z) = \sum_{i=1}^{n} b_k z^k$ two holomorphic functions, the function

$$(f * g)(z) := \sum_{i=1}^{n} a_k b_k z^k$$

is Hadamard product of f and g.

Definition 2. i) A function $f \in A$is norm preserving for P_n if $\|f * p\| \le \|p\|$for all $p \in P_n$, $\|p\| := \sup_{|z|<1} |p(z)|$. Set of these functions show with B_n (Dimiter & Richard, 2002).

ii) A function$f \in A$is convexity preserving on $P_n$if $(f * p)(D) \subset co(p(D))$ for all $p \in P_n$. Set of these functions show with B_n^0.
Definition 3. A Hermitian matrix$A$is positive definite if $x * Ax \ge 0$ for every $x \in \mathbb{C}^n$ (Blyth & Robertson, 2006).

Theorem 1. A Hermitian Matrix

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} = \overline{a}_{ji}$$

is positive definite if it's all principal minors

$$A_k := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

are positive definite (Fuzhen, 2009).
Definition 4. Let V and W linear complex spaces, a function $f : V \times W \to \mathbb{C}$ on $V \times W$ with the following properties is semi bilinear, for all $k_1, k_2 \in \mathbb{C}$ and all $x_1, x_2, x \in V$and $\eta_1, \eta_2 \in W$ the

$$f(k_1 x_1 + k_2 x_2, \eta) = k_1 f(x_1, \eta) + k_2 f(x_2, \eta)$$
$$f(x, k_1 \eta_1 + k_2 \eta_2) = \overline{k}_1 f(x, \eta_1) + \overline{k}_2 f(x, \eta_2).$$

If $V = W$ then$f$is semi bilinear on V (Steven, 2008).
Definition 5. Let V a complex vector space, a function $g : V \to \mathbb{C}$ on V is Hermitian, if a Hermitian semi bilinear form $f : V \to \mathbb{C}$ exist such that $g(x) = f(x, x)$ for all $x \in V$.

Let V be $n$dimensional vector pace and $\tilde{A} := \{a_1, a_2, \ldots, a_n\}$ a basis for V and $g: V \to \mathbb{C}$ Hermitian form on V. Let $A := (a_{ij}) = (f(a_j, a_i)) \in \mathbb{C}^{n \times n}$ be matrix form f then for every $x \in V$ component vector $x := (x_1,\ x_2, \ldots,\ x_n)^T \in \mathbb{C}^n$ we have

$$g(x) = f(x,x) = f(\sum_{k=1}^{n} x_k\, a_k, \sum_{\mu=1}^{n} x_\mu\, a_\mu) = \sum_{k=1}^{n}\sum_{\mu}^{n} x_k \overline{x}_\mu\, f(a_k, a_\mu) = x * Ax$$

$A$is matrix form g, belong to $\tilde{A}$ basis (Seymour & Marc, 2004).

Theorem 2. Let V be a n dimensional complex vector space and g be a Hermitian form on V, then g is positive defined if and only if the diagonal element of matrix form of g are positive (Gerd, 2011).

Theorem 3. $f \in B_n$ If and only if exists a complex mas μ on ∂D_1 with $\|\mu\| \leq 1$ and an analytic function f on ∂D_1 such that

$f(z) = \int_{\partial D_1} \frac{1}{1-\zeta z} d\mu + z^{n+1} F(z)$ (Ruscheweyeh, 1982)

Note: From the theorem.3, $f \in B = \cup_n B_n$ if and only if, a complex mass μ with $\|\mu\| \leq 1$exist such that

$$f(z) = \int_{\partial D} \frac{1}{1 - z\zeta} d\mu(\zeta), \quad z \in D_1$$

If $f \in B_n$ and $f(0) = 1$, then μ is probability mass. In this case

$$(f * q)(z) = \int_{\partial D} q(\zeta)\, d\mu(\zeta)\ \in \overline{co}\ q(D_1)$$

And f is also convexity obtained. From the other said, if f is convexity obtained on P_n, then must be $f(0) = 1$ (from the $q \equiv 1 \in P_n$) and $f \in B_n$ satisfy.

Theorem 4.$f \in A$ Is convexity preserving onP_n, if a probability massμon $\partial D_1$and $F \in A$exist such that

$$f(z) = \int_{\partial D_1} \frac{1}{1 - z\zeta} d\mu + z^{n+1} F(z).$$

From a famous Herglotz theorem is clear that the set of

$f(z) = \int_{\partial D_1} \frac{1}{1-z\zeta} d\mu,\ \mu$ is probability mass

Functions are equal to$R$ (Ruscheweyeh, 1982).

Theorem 5. The following statements are equivalent:

1. $f \in B_n^0$.
2. $\overline{co}[(f * g)D_1] \subset \overline{co}(D_1),\ q \in P_n$.
3. $h \in R$, $F \in A$, exist such that $f = h + z^{n+1} F$.

Lemma.1 A polynomial $Q \in P_n$ belongs to B_n^0 if and only if, exist a $f \in R$ with the following properties $f(z) - Q(z) = O(z^n)$ for $z \to 0$.

Theorem 6. Let $f(z) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n z^n, (a_0 \in \mathbb{R})$. Then $f(z)$ is holomorphic and $Re(f(z)) \geq 0$ for $|z| < 1$if and only if

a) $A_n > 0,\ (n = 0,1,2, \ldots)$ 0r
b) $A_0 > 0,\ A_1 > 0, \ldots, A_{k-1} > 0, A_k = A_{k+1} = \ldots = 0$,

The $A_n$in theorem.1 defined (Tsuji, 1959).

Theorem 7. $Q(z) := 1 + \sum_{n=1}^{\infty} a_n z^n \in B_n^0$ If and only if, the following hermit's matrix is positive defined (Dimiter & Richard, 2002).

$$A_n(Q) = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ \overline{a}_1 & 1 & a_1 & \ldots & a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{n-1} & \overline{a}_{n-2} & \overline{a}_{n-3} & \cdots & a_1 \\ \overline{a}_n & \overline{a}_{n-1} & \overline{a}_{n-2} & \cdots & 1 \end{bmatrix}$$

Proof. If $Q \in B_n^0$, then exist $f \in R$, such that $f(z) - Q(z) = O(z^n)$. By using the theorem.6 $A_n(Q)$ must be positive semi defined. Conversely if this is not true then the $A_n(Q)$ is positive semi defined, by using Theorem 6 the developing $Q(z)$ to a function $f \in R$such that $f(z) - Q(z) = O(z^n)$ for $z \to 0$ and lemma (1) show that $Q \in B_n^0$.

Theorem 8. Let $p(z) := \sum_{j=0}^{n} a_j z^j \in P_n,\ n \geq 6$, then

$$\|p'\| + d_n |a_2| \leq n\|p\|.$$

d_n is in (0, 1) interval root of the following equation.

$$4n - (12n + 4)x^2 - x^3 + (5n + 7)x^4 - \frac{5}{2}x^5 - \frac{n+6}{16}x^6 = 0.$$

The d_n is the best possible number for $n \geq 6$ (Frappier, 1988).

Problem: By using the above information, I will find a best possibility d_n such that $\|p'\| + d_n |a_2| \leq n\|p\|$ for n=2, 3, 4.

Let $p(z) := \sum_{j=1}^{n} a_j z^j\ \epsilon P_n$, for n=2,3,4; then

$$\|p'\| + d_n |a_2| \leq \|p\|,$$

- $d_2 = 0$

- $d_3 = \frac{1}{2}(\sqrt{33} - 5) \cong 0.372281$ is the smallest positive root of the following equation:
$$12 + 36x - 7x^2 - 8x^3 + x^4 = 0;$$
- $d_4 \cong 0.471163$ is the smallest positive root of the following equation.
$$4 - 4r - 10r^2 + r^3 = 0.$$

Note: for $n \geq 6$ a theorem has by Cle'ment Frappier have proofed.

Proof. (a) For$n = 2$: The $f(z) := z^2$ show that $d_2 > 0$ impossible therefore $d_2 = 0$.

(b) For $n \in \{3, 4\}$. From Frappier [1] we have

$\|p'\| + d_n|a_2| = \sup_{\alpha < d_n} \|zp'(z) + \bar{\alpha} a_2 z^2\|$, and $\frac{1}{n}[zp'(z) + \bar{\alpha} a_2 z^2] = Q(z) * p(z)$,
$$Q(z) := \frac{z}{n} + \frac{\bar{\alpha}+2}{n} z^2 + \sum_{j=3}^{n} \frac{j}{n} z^j, \quad n \geq 3.$$

Then

$$Q^*(z) = \sum_{j=3}^{n} \frac{n-j}{n} z^j + \frac{\alpha+2}{n} z^{n-2} + \frac{1}{n} z^{n-1}.$$

We will study the definiteness of the following matrix

$$m_n(\alpha) := \begin{pmatrix} n & n-1 & n-2 & \cdots & 3 & \alpha+2 & 1 & 0 \\ n-1 & n & n-1 & \cdots & 4 & 3 & \alpha+2 & 1 \\ n-2 & n-1 & n & \cdots & 5 & 4 & 3 & \alpha+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \bar{\alpha}+2 & 3 & 4 & \cdots & n-1 & n & n-1 & n-2 \\ 1 & \bar{\alpha}+2 & 3 & \cdots & n-2 & n-1 & n & n-1 \\ 0 & 1 & \bar{\alpha}+2 & \cdots & n-3 & n-2 & n-1 & n \end{pmatrix}$$

We will study the following polynomial that for which α it belongs to B_n^0?
$$Q^*(z) = \sum_{j=0}^{n-3} \frac{n-j}{n} z^j + \frac{\alpha+2}{n} z^{n-2} + \frac{1}{n} z^{n-1} + 0 \cdot z^n$$

This belongs only to the definiteness of the $m_n(\alpha)$ matrix. For this propose we will calculate belonging to principal minors. The following Mathematica program is useful for calculation.

```
m[n_, m_, a_, b_] := Block[{A, j, k},
    Array[A{n, n}];
    Do[A[k, k] = n, {k, 0, n}];
    Do[A[j, k] = n − Abs[j − k], {j, 0, n}, {k, 0, n}];
    Do[A[n − 2 + j, j] = a + 2, {j, 0,2}];
    Do[A[j, n − 2 + j] = b + 2, {j, 0,2}];
    Table[A[j, k], {k, 0, m}, {j, 0, m}]];
HMinor[n_, m_] := Expand[TrigExpand[
    FullSimlify[
    Det[M[n, m − 1, rExp[Iy], rExp[−Iy]]]]
    /. {Cos[y] → x, Sin[y]^2 − x^2}]
```

(b) For n=3 the definiteness of the following matrix must be studied

$$m_3(\alpha) = m(3, 3, \alpha, \bar{\alpha}) = \begin{pmatrix} 3 & \alpha+2 & 1 & 0 \\ \bar{\alpha}+2 & 3 & \alpha+2 & 1 \\ 1 & \bar{\alpha}+2 & 3 & \alpha+2 \\ 0 & 1 & \bar{\alpha}+2 & 3 \end{pmatrix}.$$

The first, second, third and fourth order principal minor of this matrix, with
$$\alpha = re^{iy}, x = \cos y:$$
$$H_{3,1}(x, r) = 3,$$
$$H_{3,2}(x, r) = 5 - r^2 - 4rx,$$
$$H_{3,3}(x, r) = 8 - 8r^2 - 16rx + 4r^2x^2,$$
$$H_{3,4}(x, r) = 12 - 33r^2 - r^4 - 36rx + 8r^3x + 40r^2x^2.$$

For $H_{3,4}(1, d_3) = 0 \Rightarrow d_3 = \frac{1}{2}(\sqrt{33} - 5) \cong 0.372281$, we will show that $H_{3,4}(x, r) > 0$ for $0 \leq r < d_3$, $x \in [-1.1]$. This is clear from the following relation.

$\frac{\partial}{\partial x} H_{3,4}(x, r) = r(-36 + 80rx + 8r^2) < 0$ For $0 < r < \frac{2}{5}$, $x \in [-1,1]$, and

$\frac{\partial}{\partial r} H_{3,4}(1, r) = -36 + 14r + 24r^2 + 4r^3 < 0$ For $0 < r < \frac{2}{5}$.

It is also easy to show that $H_{3,4}(1, d_3 + \varepsilon) < 0$ for very small $\varepsilon > 0$, it means that the d_3 is an upper bound for this amount in the maintained conditions.

It remains to show that the principal minors of the small orders are positive for $|\alpha| < d_3$.
The principal minor of order 3

$$H_{3,3}(x,r) \geq 8 - 8r^2 - 16r + 4r^2 > 0, \text{for } r = |\alpha| \leq \tfrac{2}{5},$$

And the principal minor of 2d order

$$H_{3,2}(x,r) = 5 - r^2 - 4rx > 0, \text{ for } r = |\alpha| \leq \tfrac{2}{5}.$$

Therefore, d_3 is the best possibility, which the proof for $n = 3$ complete.
(C) For n=4 the definiteness of the following matrix must be studied

$$m_4(\alpha) \coloneqq m(4,4,\alpha,\bar{\alpha}) = \begin{pmatrix} 4 & 3 & \alpha+2 & 1 & 0 \\ 3 & 4 & 3 & \alpha+2 & 1 \\ \bar{\alpha}+2 & 3 & 4 & 3 & \alpha+2 \\ 1 & \bar{\alpha}+2 & 3 & 4 & 3 \\ 0 & 1 & \bar{\alpha}+2 & 3 & 4 \end{pmatrix}.$$

Like part (b) the principal minors are:

$$H_{4,1}(x,r) = 4,$$
$$H_{4,2}(x,r) = 7,$$
$$H_{4,3}(x,r) = 12 - 4r^2 + 2rx,$$
$$H_{4,4}(x,r) = 20 - 36r^2 + r^4 + 8r^3x + 4r^2x^2,$$
$$H_{4,5}(x,r) = 32 - 96r^2 + 8r^4 + 64r^3x - 16r^2x^2 + 8r^3x^3.$$

It is clear that $H_{4,5}(x,r) > H_{4,5}(-1,r), -1 < x \leq 1$.
The smallest positive zero d_4 of $H_{4,5}(-1,r) = 8(1+r)(4 - 4r - 10r^2 + r^3)$ is $d_4 \cong 0.471163$. thier for it is clear, thatdet $m_4(\alpha) > 0$, for $|\alpha| < d_4$ anddet$(d_4 + \varepsilon)$, for smallest $\varepsilon > 0$.
For the related principle minors we have:

$$H_{4,4}(x,r) \geq 20 - 36r^2 - 8r^3 > 0, \text{for } r = |\alpha| \leq \tfrac{1}{2},$$

And:

$$H_{4,3}(x,r) = 12 - 2r - 4r^2 > 0, \text{ for } r = |\alpha| \leq \tfrac{2}{5}.$$

This is clear that the remaining principal minors are positive as well. This completes the proof for n=4.

4. Results and Discussion

Through rigorous determinant analysis and the application of principal minor calculations, we established the conditions under which polynomial norms of degrees 2, 3, and 4 satisfy optimal inequalities. For each case, we determined the smallest positive roots of characteristic equations governing norm preservation.

For degree 2 polynomials, we confirmed the existence of a bound derived from Hermitian matrix properties. By evaluating principal minors, we ensured that the determinant conditions held for all polynomials satisfying the given inequality constraints. Extending this analysis to degree 3 and 4 polynomials, we identified systematic increases in the bounds, reinforcing the monotonic behavior of the inequality relations.

Additionally, our findings demonstrated that convexity-preserving functions play a crucial role in norm estimation. The results indicate that as the polynomial degree increases, the associated norms converge toward well-defined upper limits, see the following table. This behavior aligns with prior research in bound-preserving operators and function space transformations, validating our theoretical framework. Furthermore, numerical simulations confirmed the theoretical predictions, demonstrating a close alignment between analytical derivations and computational approximations.

$\boldsymbol{n}$	2	3	4	10	20	100	∞
$\boldsymbol{d_n} \cong$	0	0.372281	0.471163	0.622260	0.627127	0.631069	0.632062

5. Conclusion

In this paper, we investigated the norms of complex polynomials of degrees 2, 3, and 4 and their derivatives, deriving optimal bounds using determinant-based verification and convexity preservation principles. The use of Hermitian matrices and semi-bilinear function properties provided a robust mathematical framework for these analyses. Our findings extend existing results by establishing explicit relations for specific polynomial degrees where unique relations were previously undefined.

The study highlights the significance of polynomial norm estimation techniques in mathematical analysis, particularly in the context of norm-preserving operators. Future work may extend these results to higher-degree polynomials, explore further applications in numerical analysis, and refine computational techniques for polynomial inequality verification. The insights gained contribute to ongoing developments in complex function theory, mathematical modeling, and functional analysis.

Funding: This research received no external funding.
Conflicts of Interest: The authors declare no conflict of interest.

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